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Convex Polyhedra Are Assembled of Pyramids

When working with preservice teachers in secondary school mathematics education, I have found that they sometimes have difficulty recalling formulas for calculating the volume of various polygonal prisms. To build their understanding at a conceptual level and make connections to the mathematics they knew, I introduced them to the concept of pyramidization. It is similar to triangulation of regular polygons in the plane, where we could find the area of the polygon as the sum of the areas of the triangles. Using the same strategy, we can also decompose a convex polyhedron into pyramids and find the total volume by adding the volumes of each of the individual pyramids that make up the convex polyhedron.

A *polyhedron* (plural *polyhedra* or *polyhedrons*) is a three-dimensional figure that consists of flat faces, straight edges, and vertices (see "Math Is Fun: Polyhedrons"). If the faces are all the same regular polygons, then the polyhedron is called *regular*. In this article, we will explore how to decompose convex regular polyhedra into polygonal pyramids to find their total volumes. Polygonal prisms and frustums (*truncated pyramids*) are pyramidized in Zadeh (2013) and motivate us to investigate the relationships between convex polyhedra and pyramids.

Before we consider decomposition of volumes, we review the analogous process for areas—triangulation. For example, see the regular pentagon in **figure 1**. The center of the pentagon, T, has been joined to each vertex to form five isosceles triangles. The area of each triangle is half the altitude, r, times the base, s—that is, $(1/2)rs$. The area of the pentagon, then, is $(5/2)rs$. (Note that the segment TD, or alternatively its length r, is called the apothem and is the same as the radius of the incircle of the regular polygon.)

Returning to three dimensions, we consider the regular-polygon faces of the polyhedra and assume that they have side length s. We also assume that we can inscribe a sphere of center C and radius R in all the regular polyhedra (Zazkis, Zinitsky, and Leikin 2013). This inscribed sphere will be tangent to each face at its center, T. Note that s and R as constants of the polyhedra are easy to describe and use in the volume formulas. To find the volumes, we have to introduce two other constants, such as α and β, as is done in Zazkis, Zinitsky, and Leikin (2013), where α is the central angle of the regular polygonal faces and β is the angle that R makes with the edge of the pyramid built on the faces.

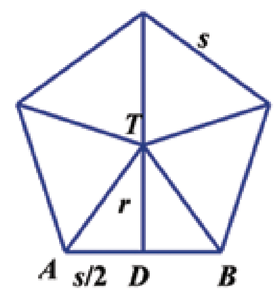

Fig. 1 A polygon is decomposed into triangles to formulate its area.

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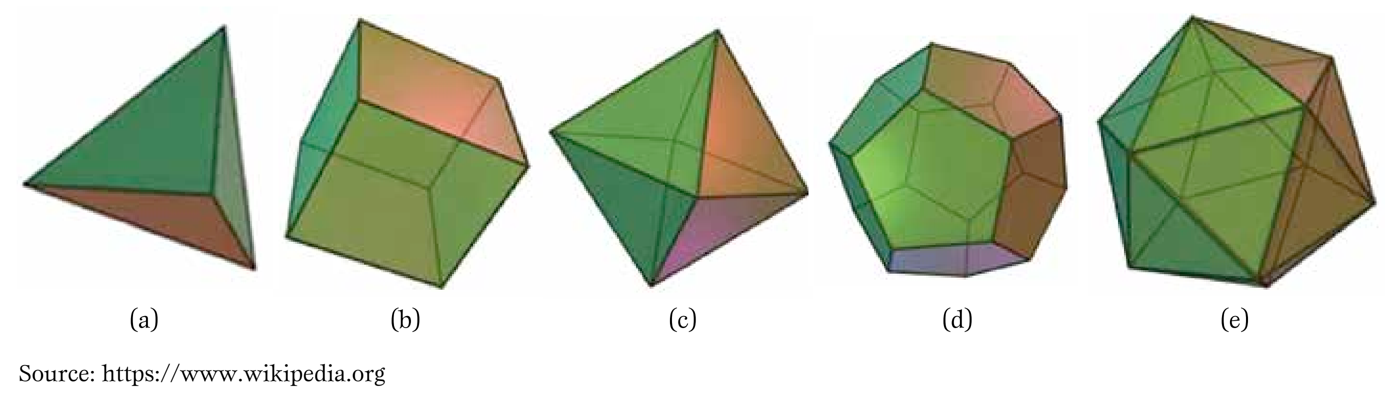

Fig. 2 The Platonic solids are regular polyhedra: tetrahedron (**a**); hexahedron (**b**); octahedron (**c**); dodecahedron (**d**); and icosahedron (**e**).

REGULAR POLYHEDRA

Geometers have studied the mathematical beauty and symmetry of the Platonic solids for thousands of years. These regular polyhedra—tetrahedron, hexahedron (cube), octahedron, dodecahedron, and icosahedron—are named after the numbers of faces (see **fig. 2**). In a *Platonic solid*, each face is the same regular polygon and the same numbers of polygons meet at each vertex. The center, C, of the polyhedron or its inscribed sphere joins to each vertex to form polygonal pyramids on the faces. For example, if the polyhedron is a cube, the pyramidization is shown in **figure 3**. Six pyramids are used, one on each face—upper, lower, left, right, front, and back.

Let's investigate the volumes for the Platonic solids as the number of faces increases.

Tetrahedrons

A tetrahedron has four equilateral triangular faces of side length s and area $A = \left(\sqrt{3}/4\right)s^2$. The inscribed sphere in the tetrahedron is tangent to each face at its center, T. On each triangular face, we build a triangular pyramid by joining the center of the sphere to each vertex (see **fig. 4**). The radius of the inscribed sphere, R, serves as the vertical height of the triangular pyramid. The volume of the pyramid is 1/3 the vertical height, R, times the area of the base, A:

$$v = \frac{1}{3}R\left(\frac{\sqrt{3}}{4}s^2\right) = \frac{\sqrt{3}}{12}Rs^2$$

A tetrahedron is made up of four of these pyramids, one on each face. Hence, the volume of the tetrahedron is

$$(1) \qquad V = 4v = 4\left(\frac{\sqrt{3}}{12}Rs^2\right) = \frac{\sqrt{3}}{3}Rs^2.$$

Hexahedrons (Cubes)

A hexahedron, the familiar cube, has six square faces of side length s and area $A = s^2$. On each square face, we build a square pyramid of vertical height $R = (1/2)s$, the radius of the inscribed sphere

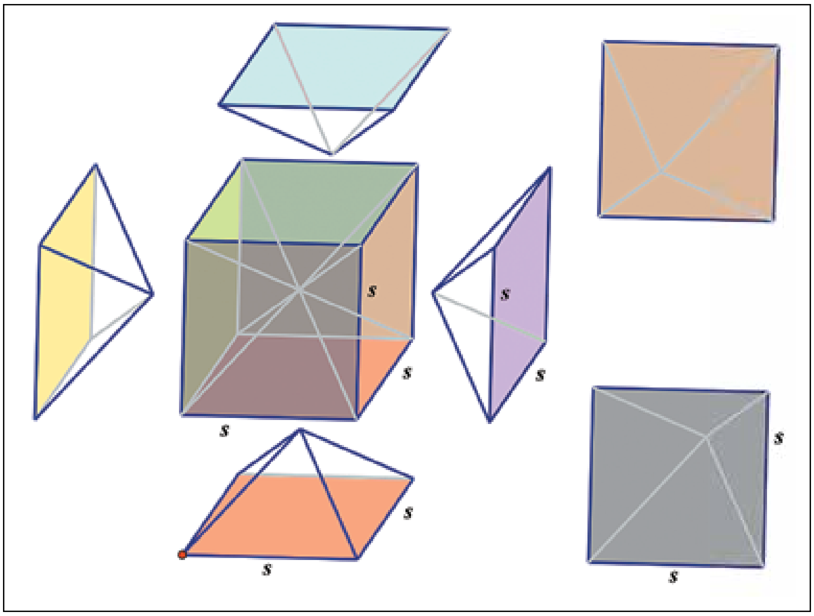

Fig. 3 A cube is decomposed into six square pyramids.

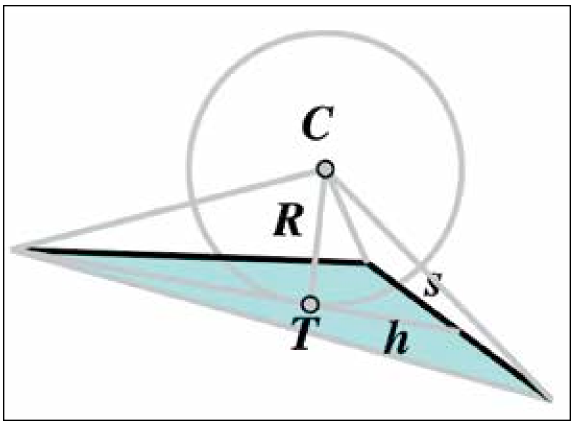

Fig. 4 The inscribed sphere is tangent to each face at its center.

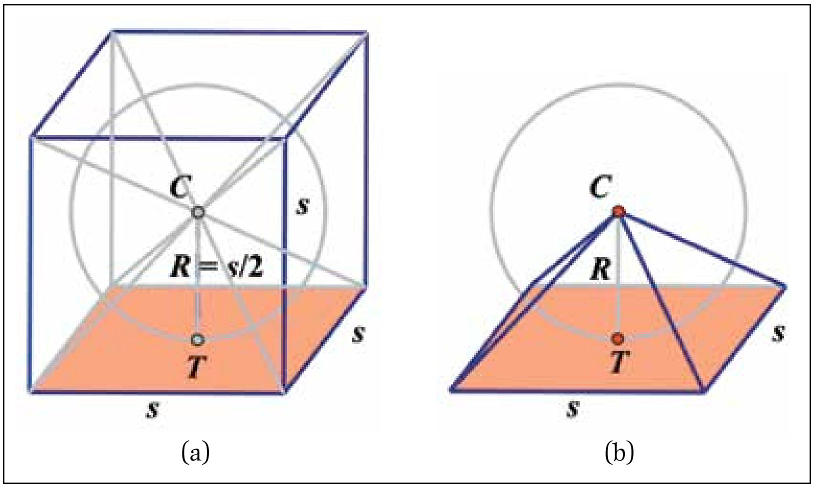

Fig. 5 The radius of a cube's inscribed sphere is the height of a constituent pyramid.

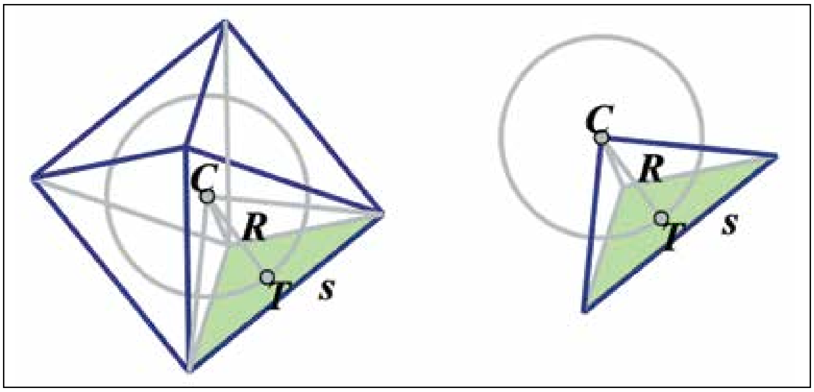

Fig. 6 The octagon's volume will be calculated on the basis of eight triangular pyramids.

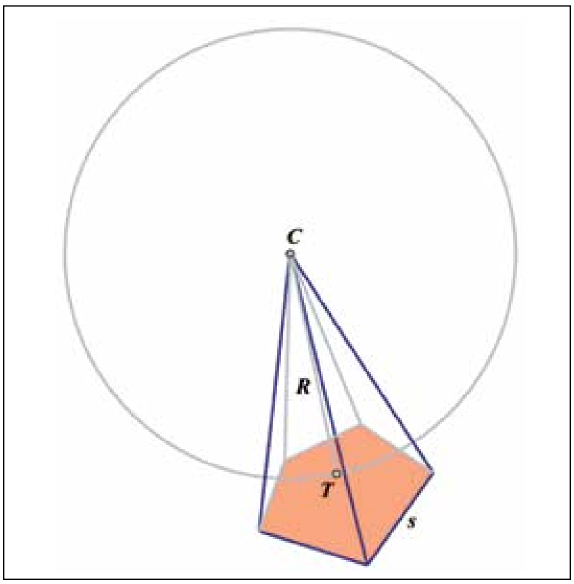

Fig. 7 The twelve constituent pyramids of a dodecahedron are pentagonal.

(see **fig. 5**). Its volume, v, is

$$v = \frac{1}{3}RA = \frac{1}{3}\left(\frac{1}{2}s\right)\cdot s^2 = \frac{1}{6}s^3.$$

A hexahedron is made up of six of these pyramids, so the volume of the cube with side length s is the familiar s^3:

$$(2) \qquad V = 6v = 6\left(\frac{1}{6}s^3\right) = s^3$$

Octahedrons

With eight equilateral triangular faces, each of side length s, the octahedron is decomposed into eight triangular pyramids of vertical height R, the radius of the inscribed sphere (see **fig. 6**). The volume of each triangular pyramid is, again, $v = \left(\sqrt{3}/12\right)Rs^2$. An octahedron, made up of eight such pyramids, has volume

$$(3) \qquad V = 8v = 8\left(\frac{\sqrt{3}}{12}Rs^2\right) = \frac{2\sqrt{3}}{3}Rs^2.$$

Dodecahedrons

A dodecahedron, which has twelve regular pentagonal faces, each of side length s, and apothem (the radius of the inscribed circle) r (see **fig. 7**). Earlier, we found the area of a regular pentagon to be $A = (5/2)rs$. Later, we find r in terms of s, so that all the volumes will be functions of R and s.

Joining the dodecahedron's center to each vertex builds twelve pentagonal pyramids with vertical height R, the radius of the inscribed sphere. The volume of each pyramid is

$$v = \frac{1}{3}RA = \frac{1}{3}R\left(\frac{5}{2}rs\right) = \frac{5}{6}Rrs,$$

so the volume of the dodecahedron is

$$(4) \qquad V = 12v = 12\left(\frac{5}{6}Rrs\right) = 10Rrs.$$

We can find r in terms of s as follows. The central angles of the pentagon are each $360°/5 = 72°$. The pentagon decomposes into five isosceles triangles, as shown in **figure 1,** with angles of 72°, 54°, and 54°. The altitude, r, is one leg of the 54-36-90° right triangle ATD. Therefore, $\tan 36° = (s/2)/r$, or

$$r = \frac{s}{2}\cdot\frac{1}{\tan 36°} = \frac{1}{2}s\cdot\cot 36°.$$

If we substitute this expression for r in formula (4), we get

$$V = 10R\left(\frac{1}{2}s \cdot \cot 36^{\circ}\right)s = 5Rs^2 \cot 36^{\circ}. \quad (5)$$

Icosahedrons

With twenty equilateral triangular faces, each of side length s, the icosahedron can be decomposed into twenty triangular pyramids of vertical height R, the radius of the inscribed sphere. The volume, v, of each triangular pyramid (as in the section on tetrahedrons) is

$$v = \frac{1}{3}R\left(\frac{\sqrt{3}}{4}s^2\right) = \frac{\sqrt{3}}{12}Rs^2.$$

Therefore, the volume of the icosahedron is

$$V = 20v = 20\left(\frac{\sqrt{3}}{12}Rs^2\right) = \frac{5\sqrt{3}}{3}Rs^2. \quad (6)$$

STRENGTHENING STUDENTS' VISUALIZATION

Instead of having students memorize various formulas to find the volume of different convex polyhedra, this article emphasizes how to find the volume at a conceptual level by decomposing the polyhedra into pyramids. Memorizing a formula for the volume of a solid does not mean that students have conceptual understanding. All students should develop the strategies and understanding to decompose a solid into small pieces to find its volume, thereby strengthening their visualizations of three-dimensional figures.

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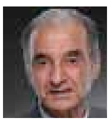

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